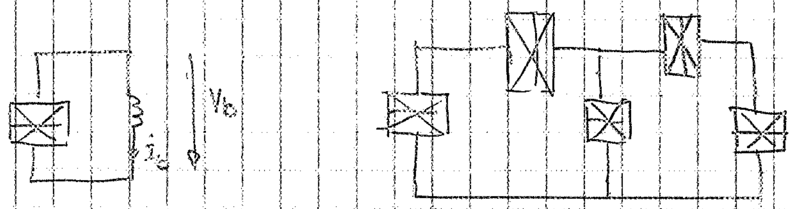
quantizing superconducting circuits

circuit made up of capacitors, inductors, Josephson junctions



branch currents $\quad i_b$

branch voltage $\quad V_b$

Kirchhoff's laws

$$\sum_{node} i_b = 0$$

$$\oint_{loop} V_b = 0$$

procedure to find Hamiltonian

write Lagrange-function

$$\mathcal{L}(q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n) = T - V$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

- conjugate momentum $p_i(q_i, \dot{q}_i, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

- invert the equations to find $\dot{q}_i$

- find Hamilton function with a Legendre transformation

$$\mathcal{H}(q_i, p_i, t) = \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}$$

$$= \sum_i \dot{q}_i p_i - \mathcal{L}$$

Hamilton's equations

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

- replace variables by operators

$$q_i \to \hat{q}_i$$

$$p_i \to \hat{p}_i \qquad [\hat{q}_i, \hat{p}_i] = i\hbar$$

$\mathcal{H} \to \hat{\mathcal{H}}$ Hamiltonian

Note: Circuit has not unique Hamitonian
depends on representation

recipe to directly find Hamiltonian

- chose one node as ground
- define a loop-free spanning tree
  each node is linked to ground
  by one and only one path
- define node voltage V_n as
  the sum of branch voltages V_b to ground
- define node current i_n as the
  current flowing to the node by the
  capacitors only

dynamical variables

node flux $\phi_n = \int_{-\infty}^{t} V_n(\tau) d\tau$

node charge $q_n = \int_{-\infty}^{t} i(\tau) d\tau$

Using Kirchhoff's laws, express branch
flux/charge as as linear combination of
node variables

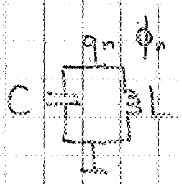
- sum energies of all branches to get
  Hamilton function

$\phi_n \longrightarrow \hat{\phi}_n$

$q_n \longrightarrow \hat{q}_n$

$$[\hat{\varphi}_n, \hat{q}_m] = i\hbar\, \delta_{nm}$$

1st example: LC oscillator



$i_C = C \frac{d}{dt} V_C \qquad \frac{di_L}{dt} L = V_L$

Kirchhoff: $\quad i_C + i_L = 0 \qquad V_C = V_L$

$\phi_n = \int V\,dt = L\,i_L = LI$

$q_n = \int i\,dt = C\,U$

$$\mathcal{L}(q,\dot q) = \frac{q^2}{2C} - \frac{L}{2}\dot q^2$$

$$0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) - \frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\left(L\dot q\right) + \frac{q}{C} = -L\ddot q - \frac{1}{C}q$$

$$\mathcal{H}(q,\phi) = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}\frac{\phi^2}{L}$$

$$\hat{\mathcal{H}}(\hat q,\hat\phi) = \frac{1}{2}\frac{\hat q^2}{C} + \frac{1}{2}\frac{\hat\phi^2}{L} \quad \text{harmonic oscillator}$$

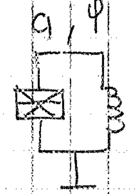
$$[\hat q, \hat\phi] = i\hbar$$

$$\hat q = \sqrt{\frac{\hbar}{2Z_c}}\left(a^\dagger + a\right) \qquad \hat a = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c\hat q + i\hat\phi\right)$$

$$\hat\phi = i\sqrt{\frac{\hbar Z_c}{2}}\left(a^\dagger - a\right) \qquad \hat a^\dagger = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z\hat q - i\hat\phi\right)$$

$$\hat{\mathcal{H}} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad Z_c = \sqrt{\frac{L}{C}}$$

RF-SQUID



$$\mathcal{H}(q, \varphi) = \mathcal{H}_{ind} + \mathcal{H}_{Jj}$$

$$\mathcal{H}_{ind} = \frac{\phi^2}{2L}$$

$$\mathcal{H}_{Jj} = \frac{q^2}{2C} - E_J \cos(\delta)$$

$$V_{Jj} = \frac{\phi_0}{2\pi} \dot{\delta}$$

$$\phi = \int_{ind} V_{ind} \, d\tau$$

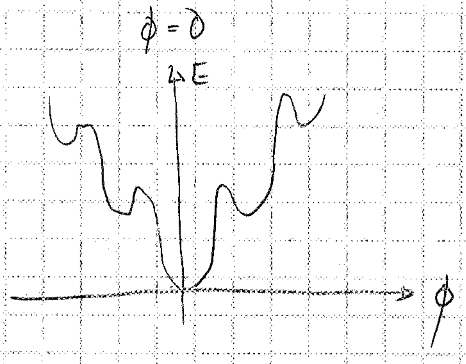
$$\phi_{ext} = \oint V = \phi - \underbrace{\int V_{Jj} \, d\tau}_{\frac{\phi_0}{2\pi} \delta}$$

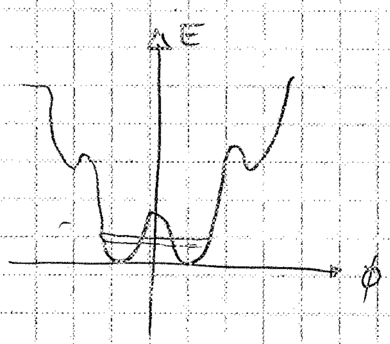
$$\mathcal{H} = \frac{q^2}{2C} + \frac{\phi^2}{2L} - E_J \cdot \cos\left(\frac{2\pi}{\phi_0} (\phi - \phi_{ext}) \right)$$

problem has 2 parameters

E_J/E_C $\quad \frac{L_J}{L}$

no analytical solution

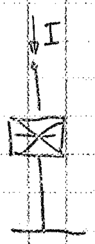




problem: qubit very sensitive to flux noise

$\boxed{\text{Slides}}$

Phase Qubit



current biased junction

$$\mathcal{H} = \frac{q^2}{2C} - I\phi - I\phi_0 \cos\left(\phi \frac{2\pi}{\phi_0}\right)$$

$$\delta = 2\pi \frac{\phi}{\phi_0} \qquad p = 2eq$$

$$\mathcal{H} = 4E_c\, p^2 - I\phi_0\,\delta - I\phi_0 \cos(\delta)$$

$$[\delta, p] = i$$

for $I \lesssim I_0$

$$U(\delta) = \phi_0 (I_0 - I)(\delta - \pi/2) - \frac{I_0\phi_0}{2\pi}(\delta - \pi/2)^3$$

$$\omega_p = \frac{1}{\sqrt{L_J(I)\,C}} = \frac{1}{\sqrt{L_J\, C_J}} \left(1 - (I/I_0)^2\right)^{1/4}$$